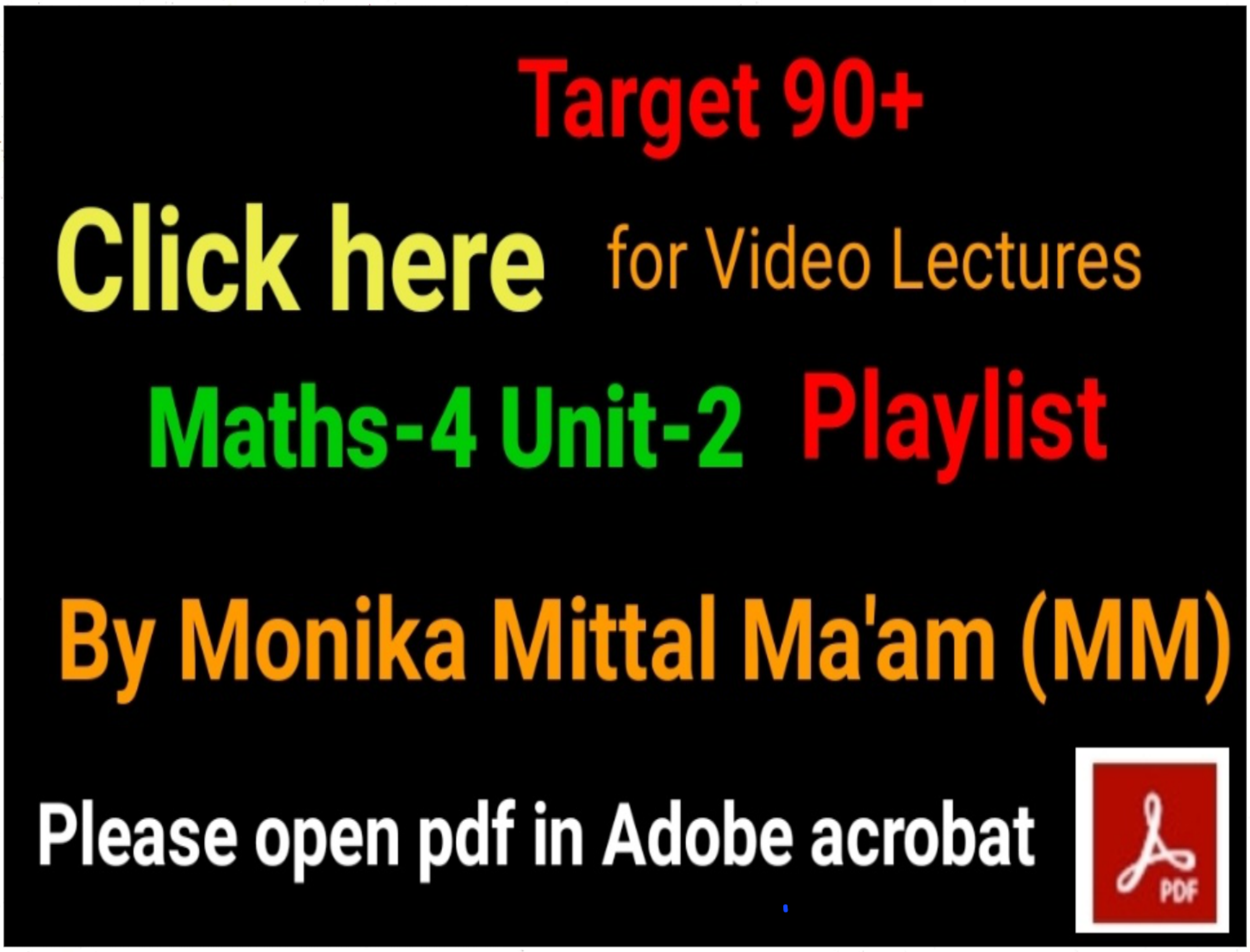
Maths-4 Unit-2 Video Playlist



(BASED ON AKTU)

Unit-2 Math-4 L-1 $\boxed{L\text{-}L\text{-}L}$

$\boxed{\text{Application of PDE}}$ $\boxed{MM\text{-}1}$

Topic: Classification of PDE of second order

Standard form

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + f\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) = 0 \qquad u = u(x,y)$$

where A, B, C are constant or continuous fn of x, y.

$u = f(x,y)$ $\quad z = z(x,y)$, $z = z(x,t)$

$$\boxed{B^2 - 4AC}$$

Now Equation is

(1) $B^2 - 4AC = 0 \Rightarrow$ Parabolic

(2) $B^2 - 4AC > 0 \Rightarrow$ Hyperbolic

(c) $B^2 - 4AC < 0 \Rightarrow$ elliptic

Note: → If A, B, C are constants then Nature of Eq^n remain same in whole region

Ex. Classify $u_{xx} + 3u_{xy} + u_{yy} = 0$ (MTU 12)

$u = u(x,y)$

$$\underset{A}{1}\frac{\partial^2 u}{\partial x^2} + \underset{B}{3}\frac{\partial^2 u}{\partial x \partial y} + \underset{C}{1}\frac{\partial^2 u}{\partial y^2}$$

$A = 1$, $B = 3$, $C = 1$ $\quad B^2 - 4AC = 9 - 4 = 5 > 0$ [Hyperbolic]

Ex. Classify $\underset{A}{4}\frac{\partial^2 u}{\partial x^2} + \underset{B}{4}\frac{\partial^2 u}{\partial x \partial t} + \underset{C}{1}\frac{\partial^2 u}{\partial t^2} = 0$ [2020-21]

$u = u(x,t)$

$A = 4$, $B = 4$, $C = 1$

$B^2 - 4AC = 16 - 4 \times 4 = 0$

$\boxed{\text{Nature is Parabolic}}$

Ex. Classify $x^2 \frac{\partial^2 u}{\partial x^2} - 1\frac{\partial^2 u}{\partial t^2} = 0$ AKTU (2019-20) $\boxed{\text{L-1-2}}$ $\boxed{\text{MM-2}}$

$B=0$
$A = x^2$
$C = -1$

$B^2 - 4AC$

$\Rightarrow 0 - 4x^2(-1)$

$= 4x^2$

$4x^2 > 0$ hyperbolic $\quad u = u(x,t)$

$x=0$ $\quad B^2-4AC = 4x^2 = 0$ Parabolic

Ex. Show that $Z_{xx} + 2xZ_{xy} + (1-y^2)Z_{yy} = 0$ is elliptic for values of x and y in region $x^2+y^2<1$, parabolic on the boundary and hyperbolic outside the region. [2018]

$Z = (x, y)$

1. $\frac{\partial^2 z}{\partial x^2} + 2x\frac{\partial^2 z}{\partial x \partial y} + (1-y^2)\frac{\partial^2 z}{\partial y^2} = 0$

$B = 2x \quad A = 1 \quad C = 1-y^2$

$B^2-4AC = 4x^2 - 4(1)(1-y^2)$

$= 4x^2 - 4 + 4y^2$

$B^2-4AC = 4(x^2+y^2-1)$

$B^2-4AC = 0 \rightarrow$ Parabolic

$4(x^2+y^2-1) = 0$

$x^2+y^2-1 = 0 \Rightarrow \boxed{x^2+y^2=1}$ ✓

$B^2-4AC > 0 \Rightarrow$ hyperbolic

$4(x^2+y^2-1) > 0$

$x^2+y^2-1 > 0 \Rightarrow \boxed{x^2+y^2>1}$ ✓

$B^2-4AC < 0 \Rightarrow$ elliptic

$4(x^2+y^2-1) < 0 \Rightarrow x^2+y^2-1 < 0$

$\boxed{x^2+y^2<1}$ ✓

Ex. Classify $(1-x^2)\frac{\partial^2 z}{\partial x^2} - 2xy\frac{\partial^2 z}{\partial x \partial y} + (1-y^2)\frac{\partial^2 z}{\partial y^2} - 2z = 0$ [AKTU 2016, 17]

Sol

$A = 1-x^2$
$B = -2xy$
$C = 1-y^2$

$B^2-4AC = 4x^2y^2 - 4(1-x^2)(1-y^2) = 4x^2y^2 - 4(1-y^2-x^2+x^2y^2)$

$B^2-4AC = 4(x^2+y^2-1)$

Parabolic if $B^2-4AC=0 \Rightarrow 4(x^2+y^2-1) = 0 \Rightarrow \boxed{x^2+y^2=1}$

Elliptic if $B^2-4AC<0 \Rightarrow (x^2+y^2-1)<0 \Rightarrow \boxed{x^2+y^2<1}$

Hyperbolic if $B^2-4AC>0 \Rightarrow (x^2+y^2-1)>0 \Rightarrow \boxed{x^2+y^2>1}$

Topic: → Method of Separation of Variables to Solve PDE

Ex. Solve by separation of Variables. [2018]

(A) — $\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x} - 2u$, $u(x,0) = 10e^{-x} - 6e^{-4x}$

$u = u(x,t)$

$u = X(x).T(t)$ — (1)

Diff eqⁿ (1) w.r.t 't', x as a const.

$\frac{\partial u}{\partial t} = X\frac{dT}{dt} = XT'$,

Diff eqⁿ (1) w.r.t 'x', t as a const

$\frac{\partial u}{\partial x} = T\frac{dX}{dx} = TX'$

Eqⁿ (A) , $XT' = TX' - 2XT$

$\boxed{\text{Divide by } XT}$, $\frac{\cancel{X}T'}{\cancel{X}T} = \frac{\cancel{T}X'}{X\cancel{T}} - \frac{2\cancel{XT}}{\cancel{XT}}$

$\boxed{\frac{T'}{T} = \frac{X'}{X} - 2 = k}$, $p^2, -p^2$

$\frac{T'}{T} = k \Rightarrow T' = kT$

$\frac{dT}{dt} = kT$

$\Rightarrow DT = kT$

$\Rightarrow (D-k)T = 0$

A.E $D \to m$

$m - k = 0$

$m = k$

$CF = c_1e^{kt}$

$\boxed{T = c_1e^{kt}}$

$\frac{X'}{X} - 2 = k \Rightarrow \frac{X'}{X} = k+2$

$X' = (k+2)X$

$\frac{dX}{dx} = (k+2)X$

$[D - (k+2)]X = 0$

AE $m - (k+2) = 0$

$m = k+2$ $CF = C_2e^{(k+2)x}$

$\boxed{X = C_2e^{(k+2)x}}$

$\boxed{u = \underbrace{C_2e^{(k+2)x}}_{X} . \underbrace{c_1e^{kt}}_{T} = c_1c_2e^{(k+2)x+kt}}$

Most general Solⁿ $\boxed{u(x,t) = \sum_{n=1}^{\infty} b_n e^{(k+2)x+kt}}$ — (2)

$$\boxed{u(x,0) = 10e^{-x} - 6e^{-4x}}$$
$\hookrightarrow t=0$

MM-2

L-2-2

Put $t=0$ in eq^n (2)

$$u(x,0) = \sum b_n e^{(k+2)x}$$

$$10e^{-x} - 6e^{-4x} = \sum_{n=1}^{\infty} b_n e^{(k+2)x}$$

$$10e^{-x} - 6e^{-4x} = b_1 e^{(k+2)x} + b_2 e^{(k+2)x} + b_3 e^{(k+2)x} + \cdots$$

$b_1 = 10$ | $b_2 = -6$ | $b_3 = b_4 = b_5 \cdots = 0$

$-1 = k+2$ | $-4 = k+2$

$k = -3$ | $k = -6$

Eq^n (2) $u(x,t) = b_1 e^{(k+2)x+kt} + b_2 e^{(k+2)x+kt} + b_3 e^{(k+2)x+kt} + \cdots$

$$u(x,t) = 10e^{(-3+2)x-3t} + (-6)e^{(-6+2)x-6t}$$

$$\boxed{u(x,t) = 10e^{-x-3t} - 6e^{-4x-6t}}$$

Contd...

L-3 (Unit-2) Math-4

Method of separation of Variable

MM-1

L-3-1

Ex. Solve by Method of Separation of Variables

$$\frac{\partial^2 u}{\partial x \partial t} = e^{-t}\cos x$$

given that $u=0$, when $t=0$ and — (A) [2013]

$\frac{\partial u}{\partial t} = 0$, when $x=0$ — (B)

$\boxed{u = u(x,t)}$

Sol^n let $u = X(x).T(t)$ — (1)

eq^n (1) w.r.t 't', x as const

$$\frac{\partial u}{\partial t} = X\frac{dT}{dt} = XT'$$

Again diff w.r.t 'x', t as const

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t}\right) = T'\frac{dX}{dx}$$

$$\frac{\partial^2 u}{\partial x \partial t} = T'X'$$

$$T'X' = e^{-t}\cos x$$

$$\boxed{\frac{T'}{e^{-t}} = \frac{\cos x}{X'} = k}$$

$$\frac{T'}{e^{-t}} = k \Rightarrow \frac{dT}{dt} = ke^{-t}$$

$$dT = ke^{-t}dt$$

$$\boxed{T = -ke^{-t} + C_1}$$

$$\frac{\cos x}{X'} = k$$

$$X' = \frac{\cos x}{k} \Rightarrow \frac{dX}{dx} = \frac{\cos x}{k}$$

$$dX = \frac{\cos x}{k}dx$$

$$\boxed{X = \frac{\sin x}{k} + C_2}$$

$$\boxed{u(x,t) = \left(\frac{\sin x}{k} + C_2\right)\left(-ke^{-t} + C_1\right)} \quad \text{— (2)}$$

→ sol^n

Put $t=0$ then $u=0$

$$u(x,0) = \left(\frac{\sin x}{k} + C_2\right)(-k.1 + C_1)$$

$$0 = \left(\frac{\sin x}{k} + C_2\right)(-k + C_1).$$

$$\frac{\sin x}{k} + C_2 \neq 0$$

$$-k + C_1 = 0$$

$$\boxed{C_1 = k}$$

$$\boxed{u(x,t) = \left(\frac{\sin x}{k} + C_2\right)\left(-ke^{-t} + k\right)} \quad \text{— (3)}$$

L-3-2

MM-2

(B) $\Rightarrow \frac{\partial u}{\partial t} = 0$ when $x=0$

eqⁿ (3) w.r.t 't'

$$\frac{\partial u}{\partial t} = \left(\frac{\sin x}{k} + c_2\right)\left(+k e^{-t}\right)$$

Apply (B), $0 = \left(\frac{\sin 0}{k} + c_2\right)\left(k e^{-t}\right)$

$$0 = c_2\left(k e^{-t}\right)$$

$\boxed{c_2 = 0}$, $k \neq 0$

Eqⁿ (3) $u(x,t) = \left(\frac{\sin x}{k} + 0\right)\left(-k e^{-t} + k\right)$

$$= \left(\frac{\sin x}{\cancel{k}}\right)\cancel{k}\left(1 - e^{-t}\right)$$

$$\boxed{u(x,t) = (\sin x)\left(1 - e^{-t}\right)}$$

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L-4 (Unit-2) Math-4

Application Of PDE.

MM-1

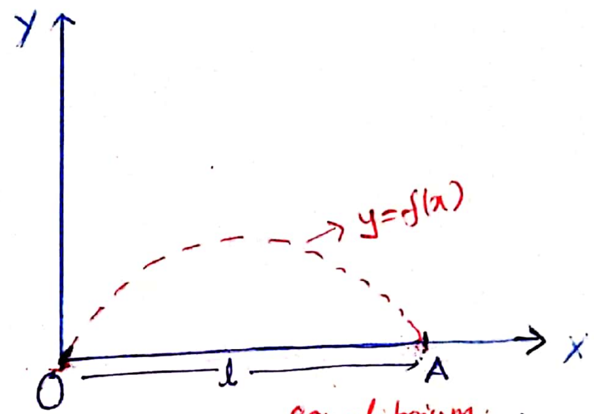
Topic: → One Dime Wave Equation

Classify One Dim wave eq^n

One Dim Wave Eq^n $$\boxed{\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}}$$ — ② $\frac{\partial^2 y}{\partial t^2} - c^2 \frac{\partial^2 y}{\partial x^2}$

$y(x,t) \rightarrow$ displacement at position 'x' and at time 't'

This PDE gives transverse Vibrations of string.



Boundary Conditions: – (depends On Ends)

At end 'O' $y=0$ at $x=0$, $y(0,t)=0$ — (i)

At end 'A' $y=0$ at $x=l$, $y(l,t)=0$ — (ii)

B.C

Initial Condition – (depend on time)

Initially, $t=0$, $y=f(x)$, $\Rightarrow y(x,0)=f(x)$. — (iv)

$\frac{dy}{dt}$ $\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0 \Rightarrow \frac{\partial y}{\partial t} = 0$ at $t=0$ — (iii)

T.C

Q → Solve $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$ wave eq^n by using separation of Variable method under boundary Condition $y=0$, at $x=0$, $y=0$ at $x=l$, and initial condition $\left.\frac{\partial y}{\partial t}\right)_{t=0} = 0$, $y=f(x)$ at $t=0$, where l is length of string.

$y = y(x,t)$

let $y = X(x)\,T(t)$ — ①

$\frac{\partial y}{\partial t} = XT'$, $\frac{\partial^2 y}{\partial t^2} = XT''$ $\qquad T' = \frac{dT}{dt}$

$\frac{\partial y}{\partial x} = TX'$, $\frac{\partial^2 y}{\partial x^2} = TX''$, $\qquad X' = \frac{dX}{dx}$

$$\frac{\cancel{X}T''}{\cancel{X}T} = c^2\frac{\cancel{T}X''}{X\cancel{T}} \;\Bigg|\; \Rightarrow \frac{T''}{T} = c^2\frac{X''}{X} \;\Bigg|\; \Rightarrow \boxed{\frac{X''}{X} = \frac{1}{c^2}\frac{T''}{T} = k}$$

MM-2 L-4-2

$$\frac{X''}{X} = k \Rightarrow X'' = kX$$

$$D^2X - kX = 0 \Rightarrow (D^2 - k)X = 0 \quad \text{— (A)}$$

$$\frac{1}{c^2}\frac{T''}{T} = k \Rightarrow T'' = c^2kT \Rightarrow D^2T - c^2kT = 0$$

$$(D^2 - c^2k)T = 0 \quad \text{— (B)}$$

Case $k = 0$

eq^n (A) $D^2X = 0$

$m^2 = 0$

$m = 0, 0$

$X = (c_1 + c_2x)e^{0x}$

$X = (c_1 + c_2x)$

eq^n (B) $D^2T = 0$

$m^2 = 0$

$m = 0, 0$

$T = c_3 + c_4t$

$y(x,t) = XT$

$$\boxed{y(x,t) = (c_1 + c_2x)(c_3 + c_4t)}$$

Case II $k = p^2$

Eq^n (A) $(D^2 - p^2)X = 0$

$m^2 - p^2 = 0$

$m = \pm p$

$X = c_1e^{px} + c_2e^{-px}$

eq^n B

$(D^2 - c^2p^2)T = 0$

$m^2 - c^2p^2 = 0$

$m = \pm cp$

$T = c_3e^{cpt} + c_4e^{-cpt}$

$$\boxed{y(x,t) = (c_1e^{px} + c_2e^{-px}) \cdot (c_3e^{cpt} + c_4e^{-cpt})}$$

Case III $k = -p^2$

eq^n (A) $(D^2 + p^2)X = 0$

$m^2 + p^2 = 0$

$m = 0 \pm pi$

$X = e^{0x}[c_1\cos px + c_2\sin px]$

eq^n B

$(D^2 + c^2p^2)T = 0$

$m^2 + c^2p^2 = 0$

$m = 0 \pm cpi$

$T = e^{0t}[c_3\cos cpt + c_4\sin cpt]$

$$\boxed{y(x,t) = (c_1\cos px + c_2\sin px)(c_3\cos cpt + c_4\sin cpt)}$$

Sol^n

$$\boxed{y(x,t) = (c_1\cos px + c_2\sin px)(c_3\cos cpt + c_4\sin cpt)} \quad \text{— (2)}$$

Apply B.C (i) $y = 0$ at $x = 0$

$0 = (c_1 + 0)(c_3\cos cpt + c_4\sin cpt)$

$$\boxed{c_1 = 0}$$

$y(x,t) = (c_2\sin px)(c_3\cos cpt + c_4\sin cpt)$ — (3)

Apply B.C (ii), $y = 0$ at $x = l$ in eq^n (3)

$0 = (c_2\sin pl)(c_3\cos cpt + c_4\sin cpt)$

$c_2\sin pl = 0$ $\qquad c_2 \neq 0$, $\sin pl = 0 = \sin n\pi$

$pl = n\pi$

$$\boxed{p = n\pi/l}$$

$$y(x,t) = \left(C_2 \sin \frac{n\pi x}{l}\right)\left(C_3 \cos \frac{cn\pi t}{l} + C_4 \sin \frac{c \cdot n\pi t}{l}\right) \quad \text{--- (4)}$$

$$\frac{\partial y}{\partial t} = \left(C_2 \sin \frac{n\pi x}{l}\right)\left(-C_3 \sin \frac{cn\pi t}{l} \cdot \frac{cn\pi}{l} + C_4 \cos \frac{cn\pi t}{l} \frac{cn\pi}{l}\right)$$

$$0 = \left(C_2 \sin \frac{n\pi x}{l}\right)\left(0 + C_4 \cdot \frac{n\pi c}{l}\right)$$

$$C_4 \cdot \frac{n\pi c}{l} = 0$$

$$\boxed{C_4 = 0}$$

$$\boxed{y(x,t) = \left(C_2 \sin \frac{n\pi x}{l}\right)\left(C_3 \cos \frac{n\pi c t}{l}\right)} \quad \text{--- (5)}$$

~~Apply I.C. (iv), $y = f(x)$ at $t=0$~~

$$\cancel{f(x) = \left(C_2 \sin \frac{n\pi x}{l}\right) C_3}$$

$$y(x,t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \cos\frac{n\pi c t}{l} \quad \text{--- (6)}$$

$\left[y = f(x) \text{ at } t=0 \right]$ $\hookrightarrow$(iv)

$C_2 C_3 = b_n$

Apply I.C. (iv)

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \rightarrow \text{Half Range Sine Series.}$$

$$\boxed{b_n = \frac{2}{l}\int_0^l f(x) \sin\frac{n\pi x}{l}\, dx}$$

Sol^n $$y(x,t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \cos\frac{n\pi c t}{l}$$

where $$b_n = \frac{2}{l}\int_0^l f(x) \sin\frac{n\pi x}{l}\, dx$$

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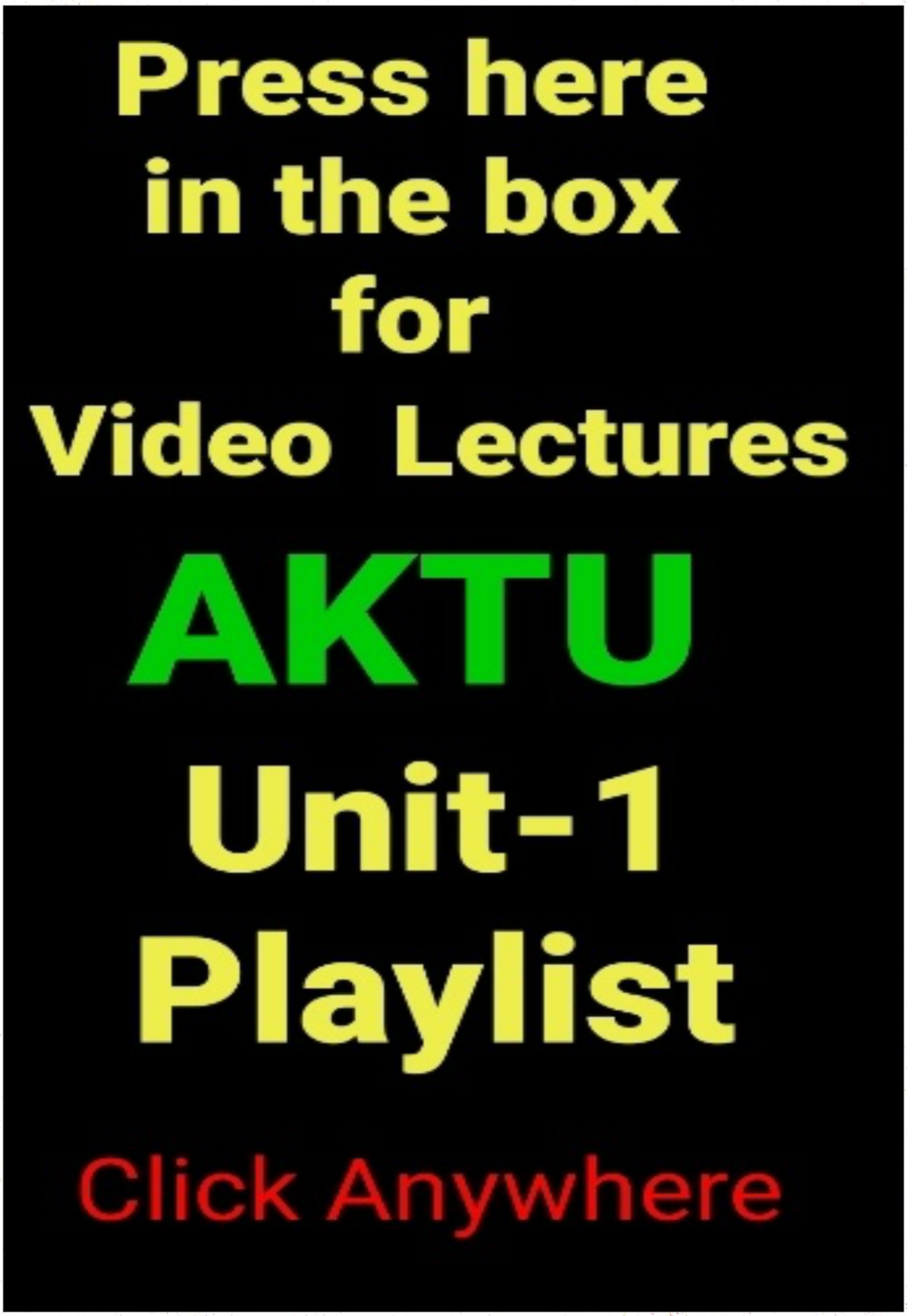
Unit-1 Playlist



L-5 (Unit-2) Math-4

MM-1

By - Monika Mittal (MM)

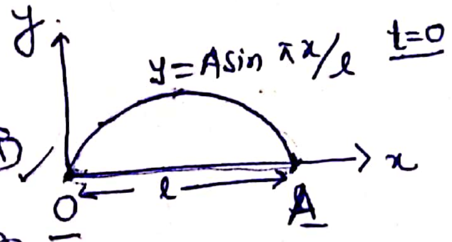
Topic: → Problems Based upon One-Dim. Wave eq^n.

Wave Eq^n $\boxed{\dfrac{\partial^2 y}{\partial t^2} = c^2 \dfrac{\partial^2 y}{\partial x^2}}$

Prob-1 A string is stretched and fastened to two points l apart. Motion is started by displacing the string in the form $y = A \sin \frac{\pi x}{l}$, from which it is released at $t=0$. Show that displacement of any point at a distance x from one end at time t is given by $y(x,t) = A \sin \frac{\pi x}{l} \cos \frac{\pi c t}{l}$

[2018, 13, 17, 11, 2012]

$\boxed{\dfrac{\partial^2 y}{\partial t^2} = c^2 \dfrac{\partial^2 y}{\partial x^2}}$ 1-D. wave eq^n.



B.C At end 'O'

$y = 0$ at $x = 0 \Rightarrow y(0,t) = 0$ — (A)

At end 'A' $y = 0$ at $x = l \Rightarrow y(l,t) = 0$ — (B)

I.C $y = A \sin \frac{\pi x}{l}$ at $t=0$ — (D)

$\frac{\partial y}{\partial t} = 0$ At = $t=0$ — (C)

Sol^n of 1-D. wave eq^n is given by.

$$y(x,t) = (C_1 \cos px + C_2 \sin px)(C_3 \cos cpt + C_4 \sin cpt) \quad \text{— (1)}$$

Apply B.C (A) $y=0$ At $x=0$

$0 = C_1 (C_3 \cos cpt + C_4 \sin cpt)$

$\boxed{C_1 = 0}$ OR $C_3 \cos cpt + C_4 \sin cpt = 0$ X

$$y(x,t) = C_2 \sin px (C_3 \cos cpt + C_4 \sin cpt) \quad \text{— (2)}$$

Apply B.C 'B' $y=0$ at $x=l$

$0 = C_2 \sin pl (C_3 \cos cpt + C_4 \sin cpt)$

$C_2 = 0$ X $\quad \sin pl = 0$, $\quad C_3 \cos cpt + C_4 \sin cpt = 0$ X

$\sin pl = \sin n\pi \Rightarrow pl = n\pi$

$\boxed{p = \frac{n\pi}{l}}$

$$y(x,t) = c_2 \sin \frac{n\pi}{l}x \left(c_3 \cos \frac{cn\pi}{l}t + c_4 \sin \frac{cn\pi}{l}t \right) \quad \text{—(3)}$$

$$\frac{\partial y}{\partial t} = c_2 \sin\left(\frac{n\pi x}{l}\right)\left[-c_3 \sin \frac{cn\pi t}{l} \frac{n\pi c}{l} + c_4 \cos \frac{cn\pi t}{l} \frac{n\pi c}{l}\right]$$

Apply I.C. (c) $\frac{\partial y}{\partial t} = 0$ at $t=0$

$$0 = c_2 \sin\left(\frac{n\pi x}{l}\right)\left[0 + c_4 \frac{n\pi c}{l}\right]$$

$$c_2 = 0 \times \qquad \boxed{c_4 = 0}$$

$$y(x,t) = c_2 \sin \frac{n\pi x}{l} \, c_3 \cos \frac{n\pi c t}{l} \quad \text{—(4)}$$

Most general soln

$$\boxed{y(x,t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \cos \frac{n\pi c t}{l}} \quad \text{—(5)}$$

Apply I.C. $y = A \sin \frac{\pi x}{l}$ at $t=0$

$$A \sin \frac{\pi x}{l} = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

$$A \sin\left(\frac{\pi x}{l}\right) = b_1 \sin \frac{\pi x}{l} + b_2 \sin \frac{2\pi x}{l} + b_3 \sin \frac{3\pi x}{l} + \cdots$$

$$b_2 = 0, \quad b_3 = 0 \ \cdots \quad b_n = 0$$

$$\boxed{b_1 = A}$$

$$y(x,t) = b_1 \sin \frac{\pi x}{l} \cos \frac{\pi c t}{l} + b_2 \sin \frac{2\pi x}{l} \cos \frac{2\pi c t}{l} + \cdots$$

$$\boxed{y(x,t) = A \sin \frac{\pi x}{l} \cos \frac{\pi c t}{l}}$$

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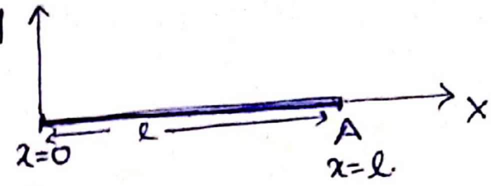
Prob-2 A tightly stretched string with fixed end points $x=0$ and $x=l$ is initially at rest in its equalibrium position. If it is set vibrating by giving to each of its points an initial velocity $\lambda x(l-x)$, find displacement of the string at any distance x from one end and at any time 't'.



I.C. $y=0$ at $t=0$ — (C)

$\frac{\partial y}{\partial t} = \lambda x(l-x)$, $t=0$ — (D)

B.C $x=0$ $y=0$ — (A)

$x=l$ $y=0$ — (B)

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$$

$$y = (c_1 \cos px + c_2 \sin px)(c_3 \cos cpt + c_4 \sin cpt) \quad \text{—①}$$

Apply 'A' $y=0$, $x=0$

$$0 = (c_1)(c_3 \cos cpt + c_4 \sin cpt)$$

$$\boxed{c_1 = 0}$$

$$y(x,t) = (c_2 \sin px)(c_3 \cos cpt + c_4 \sin cpt) \quad \text{—(2)}$$

Apply 'B' $y=0$ at $x=l$

$$0 = (c_2 \sin pl)(c_3 \cos cpt + c_4 \sin cpt)$$

$$\sin pl = 0 \Rightarrow \sin pl = \sin n\pi$$

$$pl = n\pi$$

$$\boxed{p = \frac{n\pi}{l}}$$

$$y(x,t) = \left(c_2 \sin \frac{n\pi x}{l}\right)\left(c_3 \cos \frac{cn\pi t}{l} + c_4 \sin \frac{n\pi}{l} ct\right) \quad \text{—(3)}$$

Apply I.C 'C' $y=0$, $t=0$

$$0 = \left(c_2 \sin \frac{n\pi x}{l}\right)(c_3) \Rightarrow \boxed{c_3 = 0}$$

$$y(x,t) = \left(c_2 \sin \frac{n\pi x}{l}\right)\left(c_4 \sin \frac{n\pi ct}{l}\right) \quad \text{—(4)}$$

$$\boxed{y(x,t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \sin \frac{n\pi ct}{l}} \quad \text{—(5)}$$

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$$\frac{\partial y}{\partial t} = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right) \cos \frac{n\pi c t}{l} \left(\frac{n\pi c}{l}\right)$$

Apply IC 'D' $\frac{\partial y}{\partial t} = \lambda x(l-x)$, $t=0$

$$\lambda x(l-x) = \sum_{n=1}^{\infty} b_n \left(\sin \frac{n\pi x}{l}\right) \frac{n\pi c}{l}$$

$f(x)$ $(0,l)$

$$\lambda x(l-x) = \frac{\pi c}{l} \sum_{n=1}^{\infty} (n b_n) \sin \frac{n\pi x}{l}$$

$f(x) = \sum b_n \sin \frac{n\pi x}{l}$

$b_n = \frac{2}{l}\int_0^l f(x) \sin \frac{n\pi x}{l} dx$

$$\frac{\pi c}{l} n b_n = \frac{2}{l}\int_0^l \lambda x(l-x) \sin \frac{n\pi x}{l} dx$$

$\sin n\pi = 0$

$\cos n\pi = (-1)^n$

$$= \frac{2\lambda}{l}\int_0^l \underbrace{x(l-x)}_{I} \underbrace{\sin\left(\frac{n\pi x}{l}\right)}_{II} dx$$

$$= \frac{2\lambda}{l}\left[x(l-x)\left(\frac{-\cos \frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - (l-2x)\left(\frac{-\sin \frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right) + (-2)\left(\frac{+\cos \frac{n\pi x}{l}}{\frac{n^3\pi^3}{l^3}}\right)\right]_0^l$$

$$= \frac{2\lambda}{l}\left[0 + 0 - \frac{2l^3}{n^3\pi^3}\cos n\pi - 0 - 0 + \frac{2l^3}{n^3\pi^3} 1\right]$$

$$= \frac{4\lambda}{\cancel{l}} \frac{l^{\cancel{3}2}}{n^3\pi^3}\left[1-(-1)^n\right] = \frac{4\lambda l^2}{n^3\pi^3}\left[1-(-1)^n\right]$$

$$\frac{\pi c}{l} n b_n = \frac{4\lambda l^2}{n^3\pi^3}\left[1-(-1)^n\right]$$

$$\boxed{b_n = \frac{4\lambda l^3}{n^4\pi^4 c}\left[1-(-1)^n\right]}$$

$$y(x,t) = \sum \frac{4\lambda l^3}{n^4\pi^4 c}\left[1-(-1)^n\right] \sin \frac{n\pi x}{l} \sin \frac{n\pi c t}{l}$$

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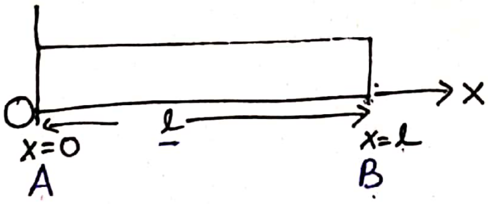
L-6 (Unit-2) $\boxed{MM-1}$

By:- Monika Mittal (MM)

Topic :→ One Dim – Heat Flow

$$\boxed{\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}} \longrightarrow 2M$$

$u = u(x,t)$ = temp at any point of bar depends on 'x' and the time 't'



Consider the flow of heat by conducting in uniform bar. It is assumed that

(a) sides of bar are insulated and loss of heat from ~~by~~ sides by conduction or radiation is negligible.

(b) temp. at all points of any cross-section is same.

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad u = u(x,t)$$

let $\quad u = X(x)\,T(t) \quad \text{—— (1)}$

$$\frac{\partial u}{\partial t} = X\frac{dT}{dt} = XT',$$

$$\frac{\partial u}{\partial x} = T\frac{dX}{dx} = TX', \quad \frac{\partial^2 u}{\partial x^2} = TX''$$

$$\Rightarrow \frac{XT'}{XT} = \frac{c^2 TX''}{XT} \Rightarrow \frac{T'}{T} = c^2\frac{X''}{X}$$

$$\Rightarrow \frac{X''}{X} = \frac{1}{c^2}\frac{T'}{T} = k$$

$$\frac{X''}{X} = k$$

$$X'' - kX = 0$$

$$(D^2 - k)X = 0 \quad \text{— (A)}$$

$$\frac{T'}{c^2 T} = k \Rightarrow T' - kc^2 T = 0$$

$$(D - kc^2)T = 0 \quad \text{—— (B)}$$

Case I $\quad k = 0$

Eq^n (A) $\Rightarrow \quad D^2 X = 0$

$$m^2 = 0 \Rightarrow m = 0, 0$$

$$CF = c_1 + c_2 x$$

$$X = c_1 + c_2 x$$

eq^n (B) $\Rightarrow \quad DT = 0$

$$m = 0$$

$$CF = c_3 e^{0t} = c_3$$

$$T = c_3$$

$$\boxed{u(x,t) = (c_1 + c_2 x)\,c_3}$$

Case II $k = p^2$

Eqⁿ (A) $(D^2 - p^2)X = 0$

$m^2 - p^2 = 0$

$m^2 = p^2$

$m = \pm p$

$CF = c_1 e^{px} + c_2 e^{-px}$

$X = c_1 e^{px} + c_2 e^{-px}$

Eqⁿ (B) $(D - p^2c^2)T = 0$

$m - p^2c^2 = 0$

$m = p^2c^2$

$CF = c_3 e^{p^2c^2t}$

$T = c_3 e^{p^2c^2t}$

$$\boxed{u(x,t) = (c_1 e^{px} + c_2 e^{-px})\, c_3 e^{p^2c^2t}}$$

Case III $k = -p^2$

Eqⁿ (A) $(D^2 + p^2)X = 0$

$m^2 + p^2 = 0$

$m = 0 \pm pi$

$CF = e^{0x}[c_1 \cos px + c_2 \sin px]$

$X = c_1 \cos px + c_2 \sin px$

$(D + p^2c^2)T = 0$

$m + p^2c^2 = 0$

$m = -p^2c^2$

$CF = c_3 e^{-p^2c^2t}$

$T = c_3 e^{-p^2c^2t}$

$$\boxed{u(x,t) = (c_1 \cos px + c_2 \sin px)\, c_3 e^{-p^2c^2t}}$$

Case I $k = 0$ $u(x,t) = (c_1 + c_2 x)c_3 = A + Bx$ (steady state)

Case II $k = p^2$ $u(x,t) = (c_1 e^{px} + c_2 e^{-px})\, c_3 e^{p^2c^2t}$ ✓ $\rightarrow e^{\infty} = \infty$

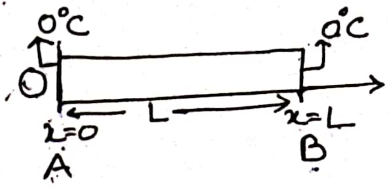
Case III $k = -p^2$ $u(x,t) = (c_1 \cos px + c_2 \sin px)\, c_3 e^{-p^2c^2t}$ ✓ $\rightarrow e^{-\infty} = 0$

$t \to \infty,\ u \to 0$

$$\boxed{u(x,t) = (c_1 \cos px + c_2 \sin px)\, c_3 e^{-p^2c^2t}}$$

$u(x,t) = A + Bx \rightarrow$ steady state

Type-I when both ends of Rod are at $0°c$ [L-6-3] [MM-3]



$$u(x,t) = u_s(x) + u_t(x,t)$$

Steady state — Transient State temp.

B.C
$u = 0°c$ at $x=0$
$u(0,t) = 0°c$ — ①
$u = 0°c$, at $x = L$
$u(L,t) = 0°c$ — ②

$$u_s = A + Bx$$

$$u_t(x,t) = (c_1 \cos px + c_2 \sin px)\, c_3 e^{-c^2p^2t}$$

$u_s = A + Bx$

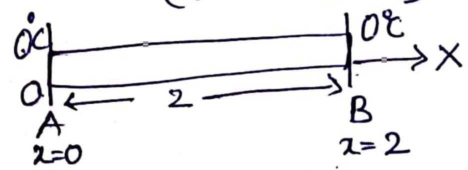
Apply ① $\left(0 = A \right.$, $u_s = Bx$, $\boxed{u_s = 0}$

Apply ② $0 = BL \Rightarrow \boxed{B=0}$

Prob – Find the temp. in a bar of length 2 whose ends are kept at zero and lateral surface insulated if initial temp is $\sin\frac{\pi x}{2} + 3\sin\frac{5\pi x}{2}$ Type I

(U.P.TU 2015)

$$u(x,t) = (c_1\cos px + c_2 \sin px)\, c_3 e^{-c^2p^2t} \quad \text{—(1)}$$



B.C At end 'A'
$u = 0°c$ at $x=0$ — Ⓐ
$u = 0°c$ at $x = 2$ — Ⓑ

I.C $u = \sin\frac{\pi x}{2} + 3\sin\frac{5\pi x}{2}$ at $t=0$ —(C)

Apply 'A'

$$0 = (c_1)\, c_3 e^{-c^2p^2t}$$

$\boxed{c_1 = 0}$ OR $c_3 = 0$ ✗

$$u(x,t) = c_2 \sin px\, c_3 e^{-c^2p^2t} \quad \text{—(2)}$$

Apply 'B'

$$0 = c_2 \sin p.2\; c_3 e^{-c^2p^2t}$$

$c_2 = 0$ ✗, $\sin 2p = 0$ ✓, $c_3 = 0$ ✗

$\sin 2p = \sin n\pi \Rightarrow 2p = n\pi$

$$\boxed{p = \frac{n\pi}{2}}$$

$$u(x,t) = c_2 \sin\left(\frac{n\pi}{2}x\right) c_3 e^{-\frac{c^2n^2\pi^2}{4}t} \quad \text{—(3)}$$

Most general solⁿ

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{2}\right) e^{-\frac{c^2n^2\pi^2 t}{4}}} \quad \text{—(4)}$$

Apply 'C'

$$\sin\frac{\pi x}{2} + 3\sin\frac{5\pi x}{2} = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{2}\right)$$

$$\sin\frac{\pi x}{2} + 3\sin\frac{5\pi x}{2} = b_1 \sin\frac{\pi x}{2} + b_2 \sin\left(\frac{2\pi x}{2}\right) + b_3 \sin\left(\frac{3\pi x}{2}\right)$$ MM-4

$$+ b_4 \sin\left(\frac{4\pi x}{2}\right) + b_5 \sin\left(\frac{5\pi x}{2}\right) + b_6 \sin\left(\frac{6\pi x}{2}\right) + \cdots$$

$\boxed{1 = b_1}$, $b_2 = 0$, $b_3 = 0$, $b_4 = 0$, $\boxed{b_5 = 3}$, $b_6 = b_7 \cdots = 0$

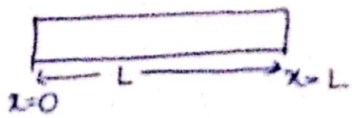
$$u(x,t) = b_1 \sin\frac{\pi x}{2} e^{-\frac{c^2\pi^2 t}{4}} + 0 + 0 + 0 + b_5 \sin\frac{5\pi x}{2} e^{-\frac{c^2 25\pi^2 t}{4}} + \cdots$$

$$\boxed{u(x,t) = 1.\sin\frac{\pi x}{2} e^{-\frac{c^2\pi^2 t}{4}} + 3\sin\left(\frac{5\pi x}{2}\right) e^{-\frac{25}{4}c^2\pi^2 t}}$$

By-Monika Mittal (MM)

Type-2 When both ends of rod are Perfectly insulated

Since ends are insulated, no heat can pass from either sides. So in this case B.C. are



$$\frac{\partial u}{\partial x} = 0 \;, x = 0$$
$$\frac{\partial u}{\partial x} = 0 \;, x = L$$

$U_s = A + Bx$

$\frac{\partial U_s}{\partial x} = B$

$\boxed{0 = B}$

$U_s = A$

$$u(x,t) = \underbrace{U_s(x)}_{\text{Steady State}} + \underbrace{U_t(x,t)}_{\text{Transient State}}$$

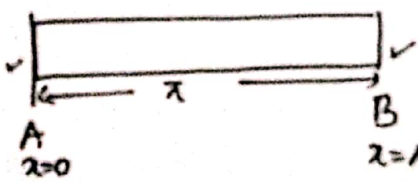
$$u(x,t) = A + (C_1 \cos px + C_2 \sin px)\, C_3 e^{-c^2p^2t}$$

Prob → The temp distribution in a bar of length π which is totally insulated at ends $x=0$ and $x=\pi$ and initial temp. distribution is $100\cos x$ find temp. distribution.

[2015]

B.C

$$\frac{\partial u}{\partial x} = 0 \;, \text{ at } x = 0 \quad \text{—(A)}$$
$$\frac{\partial u}{\partial x} = 0 \;, \text{ at } x = \pi \quad \text{—(B)}$$



I.C at $t=0$, $u = 100\cos x$

$$u(x,0) = 100\cos x \quad \text{— (C)}$$

$$u(x,t) = A + (C_1 \cos px + C_2 \sin px)\, C_3 e^{-c^2p^2t} \quad \text{—— (1)}$$

$$\frac{\partial u}{\partial x} = [-C_1 \sin px \cdot p + C_2 \cos px \cdot p]\, C_3 e^{-c^2p^2t}$$

Apply condition 'A'

$$0 = (C_2 p)\, C_3 e^{-c^2p^2t} \Rightarrow \boxed{C_2 = 0}$$

$$u(x,t) = A + C_1 \cos px \; C_3 e^{-c^2p^2t} \quad \text{—— (2)}$$

$$\frac{\partial u}{\partial x} = -c_1 \sin px \cdot p \; c_3 e^{-c^2p^2t}$$

MM-2

Apply condition 'B'

$$0 = -c_1 \sin p\pi \cdot p \; c_3 e^{-c^2p^2t}$$

$$\sin p\pi = 0 = \sin n\pi$$

$$p\pi = n\pi \Rightarrow \boxed{p = n}$$

$$u(x,t) = A + c_1 \cos nx \; c_3 e^{-c^2n^2t} \quad \text{--- (3)}$$

Most general sol^n

$$\boxed{u(x,t) = A + \sum_{n=1}^{\infty} a_n \cos nx \; e^{-c^2n^2t}} \quad \text{--- 4}$$

Apply condition 'C' $\quad u = 100\cos x$ at $t=0$

$$100\cos x = A + \sum_{n=1}^{\infty} a_n \cos nx$$

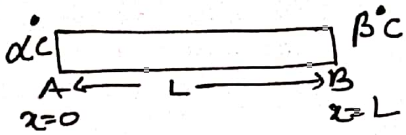
$$100\cos x = A + a_1\cos x + a_2\cos 2x + \cdots$$

$$\boxed{A=0} \qquad \boxed{a_1 = 100}, \quad a_2, a_3, a_4 \cdots = 0$$

$$u(x,t) = a_1 \cos x \, e^{-c^2t} + a_2 \cos 2x \, e^{-c^2 4t} + \cdots$$

$$\boxed{u(x,t) = 100\cos x \, e^{-c^2t}}$$

Type-3 When both ends are not at 0°C



$$u(x,t) = u_s(x) + u_t(x,t)$$

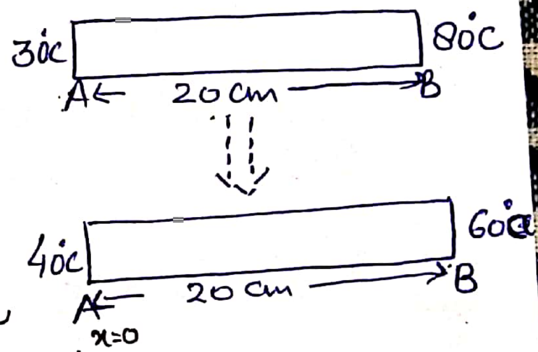
$$u_s = A + Bx$$

$$u_t(x,t) = (C_1\cos px + C_2\sin px)(C_3 e^{-c^2p^2t})$$

Prob - The ends A and B of a rod of length 20cm are at temp. 30°C and 80°C until steady state prevails. Then the temp. of rod ends are changed to 40°C and 60°C respectively. Find the temp. distribution $u(x,t)$. given $c^2 = 1$

Sol

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$



Initial temp distribution of rod is

$$= \text{temp at A} + \left(\frac{\text{Temp diff b/w A \& B}}{\text{Length of Rod}}\right)x$$

$$= T_A + \left(\frac{T_B - T_A}{L}\right)x$$

B.C

$u(0,t) = 40°C$ —(A)

$u(20,t) = 60°C$ —(B)

20 cm → 50°C

1 cm → $\frac{50}{20}$

x → $\frac{50}{20}x$

$$u(x,0) = 30 + \frac{50}{20}x$$

$$\boxed{u(x,0) = 30 + \frac{5}{2}x} \rightarrow \text{Initial Condition}$$ (C)

$$\boxed{u(x,t) = u_s + u_t}$$

$$u_s = A + Bx$$

$x=0$ $u = 40$

$$40 = A + B \cdot 0 \Rightarrow \boxed{A = 40}$$

$$u_s = 40 + Bx$$

$$60 = 40 + B \times 20 \Rightarrow 20B = 20$$

$$\boxed{B = 1}$$

$$\boxed{u_s = 40 + x}$$

$$u(x,t) = 40 + x + (c_1 \cos px + c_2 \sin px)\, c_3 e^{-c^2p^2t} \quad \text{--- (1)}$$

Apply 'A' $x=0$, $u = 40°C$

$$\cancel{40} = \cancel{40} + (c_1)\, c_3 e^{-c^2p^2t}$$

$$\underline{c_1 c_3 e^{-c^2p^2t}} = 0 \Rightarrow \boxed{c_1 = 0}$$

$$u(x,t) = 40 + x + c_2 \sin px\, c_3 e^{-c^2p^2t} \quad \text{--- (2)}$$

Apply 'B' $x = 20$, $u = 60°C$

$$\cancel{60} = \cancel{40} + \cancel{20} + c_2 \sin p.20\, c_3 e^{-c^2p^2t}$$

$$c_2 \sin p.20\, c_3 e^{-c^2p^2t} = 0$$

$$\sin p.20 = 0 = \sin n\pi$$

$$\boxed{p = \frac{n\pi}{20}}$$

$$u(x,t) = 40 + x + c_2 c_3 \sin \frac{n\pi}{20} x\, e^{-\frac{c^2 n^2 \pi^2}{400} t} \quad \text{--- (3)}$$

Most general sol^n

$$\boxed{u(x,t) = 40 + x + \sum b_n \sin \frac{n\pi x}{20}\, e^{-\frac{c^2n^2\pi^2}{400}t}} \quad \text{--- (4)}$$

Apply I.C. $u(x,0) = 30 + \frac{5}{2}x$

$$30 + \frac{5}{2}x = 40 + x + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{20}$$

$$\underline{\frac{3}{2}x - 10} = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{20} \qquad \text{[Half Range Sine series]}$$

$$b_n = \frac{2}{20}\int_0^{20} \underbrace{\left(\frac{3}{2}x - 10\right)}_{I} \underbrace{\sin \frac{n\pi x}{20}}_{II} dx$$

$$= \frac{1}{10}\left[\left(\frac{3}{2}x - 10\right)\left(\frac{-\cos \frac{n\pi x}{20}}{\frac{n\pi}{20}}\right) - \left(\frac{3}{2}\right)\left(\frac{-\sin \frac{n\pi x}{20}}{\frac{n^2\pi^2}{400}}\right)\right]_0^{20}$$

$$= \frac{1}{10}\left[-20 \cdot \frac{20}{n\pi} \cos n\pi - 10 \times \frac{20}{n\pi}\right] = \frac{1}{\cancel{10}} \times \cancel{10} \times \frac{20}{n\pi}\left[-1 - 2(-1)^n\right]$$

$$\boxed{b_n = -\frac{20}{n\pi}\left[1 + 2(-1)^n\right]}$$

$Eq^n(4)$ $\quad u(x,t) = 40 + x + \sum -\frac{20}{n\pi}\left[1+2(-1)^n\right] \sin\frac{n\pi x}{20}\, e^{-c^2\frac{n^2\pi^2}{400}t}$

$c^2=1$

L-8 (Unit-2)

By- Monika Mittal (MM)

Topic :→ Laplace Equation in 2-Dimension

Heat Equation in 2-D (2 D)

$$\boxed{\frac{\partial u}{\partial t} = c^2\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right]} \rightarrow (2M)$$

$$\boxed{1\text{ D} \quad \frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}}$$

In steady state. $\boxed{\frac{\partial u}{\partial t} = 0}$

Eq^n will become $\boxed{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0} \Rightarrow$ Laplace Eq^n

Sol^n of Laplace Eq^n by Separation of Variable. $\boxed{u(x,y)}$

let
$$u = X(x)Y(y) \quad -(1)$$

$$\frac{\partial u}{\partial x} = Y\frac{dX}{dx} = YX', \quad \frac{\partial^2 u}{\partial x^2} = YX''$$

$$\frac{\partial u}{\partial y} = X\frac{dY}{dy} = XY', \quad \frac{\partial^2 u}{\partial y^2} = XY''$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \Rightarrow \frac{YX''}{XY} + \frac{XY''}{XY} = 0$$

$$\Rightarrow \frac{X''}{X} + \frac{Y''}{Y} = 0 \quad \Rightarrow \boxed{\frac{X''}{X} = -\frac{Y''}{Y} = k}$$

$$\frac{X''}{X} = k \Rightarrow X'' - kX = 0$$
$$\Rightarrow D^2X - kX = 0$$
$$\Rightarrow (D^2 - k)X = 0 \quad -(A)$$

$$-\frac{Y''}{Y} = k \Rightarrow -Y'' = kY$$
$$Y'' + kY = 0$$
$$D^2Y + kY = 0$$
$$(D^2 + k)Y = 0 \quad -(B)$$

Case I $\quad k=0$

Eq^n (A)
$$D^2X = 0$$
$$m^2 = 0$$
$$m = 0, 0$$
$$CF = (C_1 + C_2x)e^{0x}$$
$$X = C_1 + C_2x$$

Eq^n (B)
$$D^2Y = 0$$
$$m^2 = 0 \Rightarrow m = 0, 0$$
$$CF = (C_3 + C_4y)e^{0y}$$
$$Y = (C_3 + C_4y)$$

$$\boxed{u(x,y) = (C_1 + C_2x)(C_3 + C_4y)} \quad -(2)$$

Case II $k = p^2$

eqⁿ (A) $(D^2 - p^2)X = 0$

$$m^2 - p^2 = 0$$
$$m^2 = p^2 \Rightarrow m = \pm p$$
$$CF = c_1 e^{px} + c_2 e^{-px}$$
$$X = c_1 e^{px} + c_2 e^{-px}$$

Eqⁿ (B) $(D^2 + p^2)Y = 0$

$$m^2 + p^2 = 0$$
$$m^2 = -p^2 \Rightarrow m = \sqrt{-p^2}$$
$$\Rightarrow m = 0 \pm pi$$
$$CF = e^{0y}(c_3 \cos py + c_4 \sin py)$$
$$Y = [c_3 \cos py + c_4 \sin py]$$

$$u(x,y) = XY$$
$$\boxed{u(x,y) = (c_1 e^{px} + c_2 e^{-px})(c_3 \cos py + c_4 \sin py)} \quad \text{--- (3)}$$

Case III $k = -p^2$

Eqⁿ (A) $(D^2 + p^2)X = 0$

$$m^2 + p^2 = 0$$
$$m = 0 \pm pi$$
$$CF = e^{0x}[c_1 \cos px + c_2 \sin px]$$
$$X = (c_1 \cos px + c_2 \sin px)$$

Eqⁿ (B) $(D^2 - p^2)Y = 0$

$$m^2 - p^2 = 0 \Rightarrow m = \pm p$$
$$CF = c_3 e^{py} + c_4 e^{-py}$$
$$Y = c_3 e^{py} + c_4 e^{-py}$$

$$u(x,y) = XY$$
$$\boxed{u(x,y) = (c_1 \cos px + c_2 \sin px)(c_3 e^{py} + c_4 e^{-py})} \quad \text{--- (4)}$$

$$\left.\begin{array}{ll} k = 0 & u(x,y) = (c_1 + c_2 x)(c_3 + c_4 y) \\ k = p^2 & u(x,y) = (c_1 e^{px} + c_4 e^{-px})(c_3 \cos py + c_4 \sin py) \\ k = -p^2 & u(x,y) = (c_1 \cos px + c_2 \sin px)(c_3 e^{py} + c_4 e^{-py}) \end{array}\right\}$$

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Prob Use separation of Variables method to solve

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

subject to the condition

$u(0,y) = u(l,y) = u(x,0) = 0$ and $u(x,a) = \sin\frac{n\pi x}{l}$

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$u(x,y)$

Sol $u = XY$ — (1)

$k=0, \quad u(x,y) = (c_1 + c_2 x)(c_3 + c_4 y)$

$k=p^2, \quad u(x,y) = (c_1 e^{px} + c_4 e^{-px})(c_3 \cos py + c_4 \sin py)$

$k=-p^2, \quad u(x,y) = (c_1 \cos px + c_2 \sin px)(c_3 e^{py} + c_4 e^{-py})$

$u(0,y) = 0$ — (A)

$u(l,y) = 0$ — (B)

$u(x,0) = 0$ — (C)

$u(x,a) = \sin\frac{n\pi x}{l}$ — (D)

k=0

$u(x,y) = (c_1 + c_2 x)(c_3 + c_4 y)$

Apply (A) $\quad x=0 \quad u=0$

$0 = c_1 (c_3 + c_4 y)$

$\boxed{c_1 = 0}$

$u(x,y) = (c_2 x)(c_3 + c_4 y)$

Apply (B), $\quad x = l, \quad u=0$

$0 = (c_2 l)(c_3 + c_4 y)$

$\boxed{c_2 = 0}$

$\boxed{u(x,y) = 0}$ X neglect

$k=p^2$ $\quad u(x,y) = (c_1 e^{px} + c_2 e^{-px})(c_3 \cos py + c_4 \sin py)$

Apply (A) $\quad u=0, \quad x=0$

$0 = (c_1 + c_2)(c_3 \cos py + c_4 \sin py)$

$c_1 + c_2 = 0 \Rightarrow \boxed{c_2 = -c_1}$

$u(x,y) = (c_1 e^{px} - c_1 e^{-px})(c_3 \cos py + c_4 \sin py)$

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Apply (B) $x=l$, $u=0$

Gives $0 = (c_1 e^{pl} - c_1 e^{-pl})(c_3 \cos py + c_4 \sin py)$

$c_1(e^{pl} - e^{-pl}) = 0 \Rightarrow \boxed{c_1 = 0}$

$\boxed{u(x,y) = 0}$ X neglect

$\underline{K = -p^2}$ $u(x,y) = (c_1 \cos px + c_2 \sin px)(c_3 e^{py} + c_4 e^{-py})$

Apply 'A' $u=0$ at $x=0$

$0 = c_1 (c_3 e^{py} + c_4 e^{-py})$

$\boxed{c_1 = 0}$

$u(x,y) = (c_2 \sin px)(c_3 e^{py} + c_4 e^{-py})$

Apply 'B' $u=0$ at $x=l$

$0 = (c_2 \sin pl)(c_3 e^{py} + c_4 e^{-py})$

$c_2 = 0$ X OR $\boxed{\sin pl = 0} \Rightarrow \sin pl = 0 = \sin n\pi$

$pl = n\pi$

$\boxed{p = \frac{n\pi}{l}}$

$u(x,y) = c_2 \sin \frac{n\pi x}{l} \left[c_3 e^{\frac{n\pi y}{l}} + c_4 e^{-\frac{n\pi y}{l}} \right]$

Apply 'C' $u=0$, $y=0$

$0 = c_2 \sin \frac{n\pi x}{l} [c_3 + c_4] \Rightarrow c_3 + c_4 = 0 \Rightarrow \boxed{c_4 = -c_3}$

$u(x,y) = c_2 \sin \frac{n\pi x}{l} \left[c_3 e^{\frac{n\pi y}{l}} - c_3 e^{-\frac{n\pi y}{l}} \right]$

$u(x,y) = c_2 c_3 \sin \frac{n\pi x}{l} \left[e^{\frac{n\pi y}{l}} - e^{-\frac{n\pi y}{l}} \right]$

Most general sol^n $\boxed{u(x,y) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \left[e^{\frac{n\pi y}{l}} - e^{-\frac{n\pi y}{l}} \right]}$

Apply 'D' $u(x,a) = \sin \frac{n\pi x}{l}$

$$\sin\frac{n\pi x}{l} = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}\left[e^{\frac{n\pi a}{l}} - e^{-\frac{n\pi a}{l}}\right]$$

$$1 = b_n\left[e^{\frac{n\pi a}{l}} - e^{-\frac{n\pi a}{l}}\right]$$

$$\sinh x = \frac{e^{x} - e^{-x}}{2}$$

$$b_n = \frac{1}{e^{\frac{n\pi a}{l}} - e^{-\frac{n\pi a}{l}}}$$

$$= \frac{1}{2\sinh\left(\frac{n\pi a}{l}\right)}$$

$$\boxed{u(x,y) = \sum_{n=1}^{\infty} \frac{1}{2\sinh\left(\frac{n\pi a}{l}\right)} \sin\left(\frac{n\pi x}{l}\right)\left[e^{\frac{n\pi y}{l}} - e^{-\frac{n\pi y}{l}}\right]}$$